

KÄHLER-RICCI SOLITONS ON TORIC FANO ORBIFOLDS

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ABSTRACT. We prove the existence of Kähler-Ricci solitons on toric Fano orbifolds, hence extend the theorem of Wang and Zhu [WZ] to the orbifold case.

1. INTRODUCTION

A complex orbifold of dimension n is a Hausdorff space X with a family of local uniformizing charts $\{(\tilde{U}, G, \varphi)\}$. Here, \tilde{U} is an open subset of \mathbb{C}^n , G is a finite group of bi-holomorphic transformations of \tilde{U} , and φ is a continuous map from \tilde{U} to an open set $U \subset X$, such that it induces a homeomorphism $\tilde{U}/G \rightarrow U$. The notion of orbifold was first introduced by Satake ([Sa1]) in the name of “V-manifold” in the 1950’s. An orbifold is a generalization of manifold, and we can define orbifold-smooth functions and maps by requiring that the corresponding liftings to the local uniformizing charts can be extended smoothly to the whole of the charts. Many theorems on manifolds, like Hodge decomposition theorem and Kodaira imbedding theorem, were generalized to orbifolds by Baily and Satake in [Bai1], [Bai2] and [Sa2]. The analysis on orbifolds is also studied by many people, for example, see [Ch] for a discussion of Sobolev spaces.

On a complex orbifold X , one can define orbifold Kähler metrics and corresponding Ricci forms as on manifolds. The Ricci form is a closed form and hence defines a cohomology class in the Dolbeault group, and we call it the first Chern class of X , denoted by $c_1(X)$.

Definition 1.1. *A complex normal variety X with only orbifold singularities is called Fano if the Weil divisor $-K_X$ is an ample \mathbb{Q} -Cartier divisor, i.e. a multiple of $-K_X$ is ample Cartier. Equivalently, (by Baily’s embedding theorem [Bai2]) X is called a Fano orbifold if one can represent $c_1(X)$ by an orbifold Kähler form. X is called Gorenstein Fano if $-K_X$ itself is an ample Cartier divisor.*

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Like on manifolds, a fundamental problem in differential geometry of orbifolds is the existence of canonical metrics, like the Einstein metrics. In this paper, we study the existence of Kähler-Ricci solitons on Fano orbifolds.

Definition 1.2. *Let X be a Fano orbifold and ω_g be the Kähler form of a Kähler metric g on X with $\frac{1}{2\pi}[\omega_g] = c_1(X) > 0$, where*

$$\omega_g := \sqrt{-1}g_{i\bar{j}}dz^i \wedge d\bar{z}^j.$$

ω_g is called a Kähler-Ricci soliton if there is an orbifold holomorphic vector field v such that

$$(1.1) \quad Ric(\omega_g) - \omega_g = L_v \omega_g,$$

where $Ric(\omega_g)$ is the Ricci form of ω_g defined by

$$Ric(\omega_g) := -\sqrt{-1}\partial\bar{\partial} \log \det(g_{i\bar{j}}).$$

Here by “an orbifold holomorphic vector field”, we mean a holomorphic vector field v on the regular part X_{reg} , such that for any local uniformizing chart $\pi : \tilde{U} \rightarrow U \subset X$, the lifting of v extends to the whole of \tilde{U} as a holomorphic vector field on \tilde{U} , see [Bai2]. The Lie derivative of a form η with respect to a complex vector field v is defined by the Cartan formula

$$L_v \eta := di_v \eta + i_v d\eta.$$

Remark 1.3. *Since the singular set of a normal variety always has codimension at least 2, by a standard extension theorem in complex analysis, any holomorphic vector field v on the X_{reg} of a normal orbifold X is an orbifold holomorphic vector field on X .*

The Kähler-Ricci soliton is a generalization of Kähler-Einstein metric, it is also conjectured to be the limit of Kähler-Ricci flow ([T2]). The main result of this paper is the following theorem:

Theorem 1.4. *For any toric Fano orbifold X , there exists a T -invariant Kähler-Ricci soliton metric, the soliton metric is Einstein if and only if the Futaki invariant of X vanishes.*

Remark 1.5. *The uniqueness theorem of Tian and Zhu in [TZ1] and [TZ2] should also hold for Fano orbifolds. The uniqueness of Kähler-Einstein metrics on Fano orbifolds is true indeed.*

For toric Fano manifolds, the existence of Kähler-Ricci solitons was proved by Wang and Zhu in [WZ]. In [Na], Nakagawa also solved the existence problem of Kähler-Einstein metrics on toric Gorenstein Fano orbifolds in dimension 2. He also conjectures that the vanishing of Futaki invariant should be a sufficient condition for the existence of Kähler-Einstein metrics on toric

Gorenstein Fano orbifolds and in this case the automorphism group is reductive. Our Theorem 1.4 confirms a generalized version of his conjecture.

The organization of this paper is as follows. In section 2, we review relevant results on toric varieties. In section 3, we use the torus action to rewrite the soliton equation as a real Monge-Ampère equation. Finally in section 4, we prove the main theorem by establishing various a priori estimates along the same lines as [WZ]. Then we give two examples of toric Fano orbifolds.

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2. KÄHLER METRICS ON A TORIC FANO VARIETY

Let $N \cong \mathbb{Z}^n$ be a lattice, and $Q \subset N_{\mathbb{R}} \cong \mathbb{R}^n$ a convex lattice polytope, i.e., the vertices of Q are element of N . Suppose Q contains the origin in its interior, then the cones over the faces of Q form a complete fan. The toric variety X_Q associated to this fan is a normal projective variety. When all the vertices of Q are primitive and all the faces of Q are simplicial, then X_Q is a Fano orbifold. Conversely, all the toric Fano orbifolds are obtained in this way (see [Deb]). In particular, for any $n \geq 2$, there are infinitely many isomorphism classes of toric Fano orbifolds of dimension n .

Let X_Q be a toric Fano orbifold as above. Let $M := \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be the dual of N , and $P \subset M_{\mathbb{R}}$ the dual of Q defined by

$$P := \{y \in M_{\mathbb{R}} \mid \langle y, x \rangle \geq -1, \forall x \in Q\}.$$

The polytope also contains the origin in its interior, and X_Q is Gorenstein Fano if and only if P is also a lattice polytope. The isomorphism classes of toric Gorenstein Fano varieties of any dimension are finite. For dimension 2, there are exact 16 different classes, see [Na] or Chapter 8 of [CLS].

From a differential geometric viewpoint, the dual polytope P is more important. Since the faces of Q are simplicial, at every vertex of P there are precisely n facets meeting at this vertex. Hence P is a “rational simple polytope” of Lerman and Tolman [LT]. In [LT], the authors showed that for

any rational simple polytope P ,¹ there is a Kähler toric orbifold obtained by a symplectic reduction construction, and this orbifold is isomorphic to the toric variety associated to the dual fan of P (See Theorem 1.7 of [LT]). In our case, P is the dual of Q , hence the Kähler toric orbifold obtained by Lerman and Tolman is exactly X_Q . In particular, X_Q has a “canonical” Kähler metric. By the work of Guillemin and Abreu [Gu], [Ab] (see also [CDG] and [BGL]), this Kähler metric (called the “Guillemin metric”) has a nice expression using the combinatorial data of P . When X_Q is Fano, the Kähler form of the Guillemin metric is in the Dolbeault class $2\pi c_1(X_Q)$. Now we review Guillemin and Abreu’s result in our case.

In the following of this paper, we fix a lattice polytope Q as above and write $X := X_Q$. We denote the complex torus by $T_{\mathbb{C}} \cong (\mathbb{C}^*)$, which is an open dense subset of X , with the standard coordinates (z^1, \dots, z^n) . We also denote by T the maximal torus subgroup $T := \{(e^{i\theta_1}, \dots, e^{i\theta_n}) \mid \theta_i \in \mathbb{R}\}$. A T -invariant function ϕ on $T_{\mathbb{C}}$ can be viewed as a function of $x = (x^1, \dots, x^n)$, where $x^i := \log |z^i|^2$, so we can identify it as a function in \mathbb{R}^n . In this case, we have

$$\sqrt{-1}\partial\bar{\partial}\phi = \sqrt{-1} \sum_{i,j} \phi_{ij} \frac{dz^i}{z^i} \wedge \frac{d\bar{z}^j}{\bar{z}^j}.$$

In particular, when ϕ is a potential function of a Kähler metric, we have $g_{i\bar{j}} = \phi_{ij} \frac{1}{z^i} \frac{1}{\bar{z}^j}$, thus

$$\sqrt{-1}\partial\bar{\partial} \log \det(g_{i\bar{j}}) = \sqrt{-1}\partial\bar{\partial} \log \det(\phi_{ij}).$$

Let the vertices of Q be $\mathbf{n}^{(i)} \in N$, $i = 1, \dots, d$. Then we have

$$P = \{y \in M_{\mathbb{R}} \mid \langle y, \mathbf{n}^{(i)} \rangle \geq -1, i = 1, \dots, d\}.$$

In the interior of P (denoted by P^o) we define

$$(2.1) \quad l_i(y) := \langle y, \mathbf{n}^{(i)} \rangle + 1,$$

and

$$(2.2) \quad u^0 := \sum_i l_i \log l_i.$$

It is easy to check that u^0 is strictly convex in P^o , and the gradient map Du^0 is a diffeomorphism to \mathbb{R}^n . We denote the Legendre transform of u^0 by ϕ^0 , i.e.

$$\phi^0(x) = \langle Du^0(y), y \rangle - u^0(y) = \sum_i (l_i(y) - \log l_i(y)) - d,$$

¹In [LT], the authors consider “labeled rational simple polytopes”. In our case, all the labels equal to 1.

where $x \in \mathbb{R}^n$ and $y \in P^\circ$ are related by $x = Du^0(y)$. Then ϕ^0 is a strictly convex smooth function on \mathbb{R}^n , and the Guillemin metric is given by $\omega_0 = \sqrt{-1}\partial\bar{\partial}\phi^0$.

We have the following properties of the Guillemin metric, which is used in the next section.

Lemma 2.1. *We have*

$$|\log \det D^2\phi^0 + \phi^0| \leq C$$

in \mathbb{R}^n .

Proof. Note that the gradient map $D\phi^0$ is a diffeomorphism from \mathbb{R}^n to the interior of P , we can work on the polytope P . By the property of Legendre transforms, we know that for any $x \in \mathbb{R}^n$, there is a unique $y \in P$ such that $y = D\phi^0(x)$ and $x = Du^0(y)$, moreover, we have

$$\phi^0(x) = \langle Du^0(y), y \rangle - u^0(y),$$

and

$$\det(D^2\phi^0)(x) = \det(D^2u^0)^{-1}(y).$$

By (2.2), it suffices to bound $\log \det(D^2u^0) + \sum_i \log l_i$.

By (2.1) and (2.2), we have

$$(u^0)_{pq} = \sum_i \frac{\mathbf{n}_p^{(i)} \mathbf{n}_q^{(i)}}{l_i}.$$

A direct computation shows that

$$\det(D^2u^0) = \sum_{1 \leq i_1 < \dots < i_n \leq d} \frac{\det(\mathbf{n}^{(i_1)}, \dots, \mathbf{n}^{(i_n)})^2}{l_{i_1} \dots l_{i_n}}.$$

Then the lemma follows easily from this expression. \square

Lemma 2.2. *Let the vertices of P be $p^{(1)}, \dots, p^{(m)}$, and define*

$$v(x) := \max_k \{\langle x, p^{(k)} \rangle\}.$$

Then we have

$$|\phi^0 - v| \leq C$$

in \mathbb{R}^n .

Proof. We also work on the polytope P . Let $y \in P$ be the unique point such that $y = D\phi^0(x)$ and $x = Du^0(y)$. So we have

$$\phi^0(x) = \langle Du^0(y), y \rangle - u^0(y) \leq v(x) - u^0(y) \leq v(x) + C.$$

On the other hand, suppose $v(x) = \langle x, p^{(k_0)} \rangle$, then we have

$$v(x) = \left\langle \sum_i (1 + \log l_i(y)) \mathbf{n}^{(i)}, p^{(k_0)} \right\rangle,$$

thus

$$v(x) - \phi^0(x) \leq \sum_i \log l_i(y) (1 + \langle \mathbf{n}^{(i)}, p^{(k_0)} \rangle) + C' \leq C,$$

since $1 + \langle \mathbf{n}^{(i)}, p^{(k_0)} \rangle$ is nonnegative and the l_i 's are bounded from above on P . \square

3. KÄHLER-RICCI SOLITON EQUATION ON TORIC FANO ORBIFOLDS

We start with the general soliton equation. Let ω_0 be the Kähler form of the Guillemin metric $g_{i\bar{j}}^0$, and

$$\omega_g = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi,$$

where φ is a smooth function on the regular part of X such that the pull-back of ω_g on any local uniformizing chart \tilde{U} extends to a Kähler form on \tilde{U} , that is, ω_g is an orbifold Kähler form. For an orbifold holomorphic vector field v , we have

$$L_v \omega_g = L_v \omega_0 + \sqrt{-1} \partial \bar{\partial} v(\varphi).$$

By Hodge decomposition theorem on orbifolds ([Bai2]), there is a unique complex valued function θ_v such that

$$i_v \omega_0 = \sqrt{-1} \bar{\partial} \theta_v,$$

with the normalization condition $\int_X \exp(\theta_v) \omega_0^n = \int_X \omega_0^n$. From this, we have

$$L_v \omega_0 = \sqrt{-1} \partial \bar{\partial} \theta_v.$$

Let h be the Ricci potential of ω_0 , namely

$$\text{Ric}(\omega_0) - \omega_0 = \sqrt{-1} \partial \bar{\partial} h,$$

with $\int_X \exp(h) \omega_0^n = \int_X \omega_0^n$. Then the equation (1.1) becomes

$$\sqrt{-1} \partial \bar{\partial} \left(\log \frac{\det(g_{i\bar{j}}^0 + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}}^0)} - h + \varphi + \theta_v + v(\varphi) \right) = 0,$$

thus

$$(3.1) \quad \det(g_{i\bar{j}}^0 + \varphi_{i\bar{j}}) = \det(g_{i\bar{j}}^0) e^{h - \theta_v - v(\varphi) - \varphi}.$$

Now we use the special symmetry of the toric variety. Suppose ω_g is also T -invariant, then the restriction of φ to $T_{\mathbb{C}}$ can be viewed as a function in \mathbb{R}^n , so we have

$$g_{i\bar{j}} = \phi_{ij} \frac{1}{z^i} \frac{1}{\bar{z}^j},$$

where $\phi := \phi^0 + \varphi$. Let $v := \sum_i c_i z^i \frac{\partial}{\partial z^i}$, then one can also check that

$$v(\varphi) = \sum_i c_i \varphi_i$$

and

$$\theta_v = \sum_i c_i \phi_i^0 - c_v$$

for some constant c_v . Moreover, by Lemma 2.1, we know that there is a constant \tilde{c} such that

$$\log \det(D^2 \phi^0) + \phi^0 + h = \tilde{c},$$

thus we have

$$(3.2) \quad \det(\phi_{ij}) = e^{-c-\phi-\sum_i c_i \phi_i},$$

where the constant c depends only on the initial metric g^0 and the holomorphic vector field $v = \sum_i c_i z^i \frac{\partial}{\partial z^i}$.

Now we are in a position to determine the constants c_i 's.

Proposition 3.1. *The necessary condition to have a solution of (3.2) is that the c_i 's satisfy the equations*

$$(3.3) \quad \int_P y^i e^{\sum_l c_l y^l} dy = 0, \quad i = 1, \dots, n.$$

Proof. Let ϕ be a solution of (3.2). Then by (3.2) we have

$$\int_P y^i e^{\sum_l c_l y^l} dy = \int_{\mathbb{R}^n} \phi_i e^{\sum_l c_l \phi_l} \det(\phi_{ij}) dx = e^{-c} \int_{\mathbb{R}^n} \frac{\partial}{\partial x^i} (e^{-\phi}) dx = 0.$$

□

Since P contains the origin in its interior, it is easy to see that the c_i 's exist and are uniquely determined by (3.3), see, for example [WZ] or [Do]. Actually, one needs only to consider the convex function $F(s_1, \dots, s_n) := \int_P e^{\sum_l s_l y^l} dy$. Suppose $B_r(0) \subset P$ is a ball in P , and set $\Sigma_r := \{y \in B_r(0) \mid y^1 \geq \frac{1}{2}|y|\}$, then

$$F(s) \geq \int_{\Sigma_r} e^{\frac{1}{2}|s||y|} dy,$$

that is, F is proper. Hence there is a unique minimum point (c_1, \dots, c_n) of F , which satisfies (3.3).

When $c_i = 0$ for all $1 \leq i \leq n$, i.e., the barycenter of P is the origin, then the vector field $v = 0$ and the soliton equation (3.1) becomes the Kähler-Einstein equation.

Proposition 3.2. *The barycenter of P is the origin if and only if the Futaki invariant of X vanishes.*

Proof. First, we use a theorem of Cox in [Co], namely, for the toric orbifold X the maximal torus of $\text{Aut}(X)$ is exactly T . Let $\mathfrak{g}(X)$ be the Lie algebra of $\text{Aut}(X)$, consisting of holomorphic vector fields on X , and let the Cartan decomposition of $\mathfrak{g}(X)$ be

$$\mathfrak{g}(X) = \mathfrak{h}(X) + \sum_i \mathbb{C}w_i,$$

where $\mathfrak{h}(X)$ is the Lie algebra of $T_{\mathbb{C}}$, generated by $v_i = z^i \frac{\partial}{\partial z^i}$, $i = 1, \dots, n$, and the w_i 's are the common eigenvectors of the adjoint actions ad_v for $v \in \mathfrak{h}(X)$. For any w_i , there must be a $v \in \mathfrak{h}(X)$ such that $ad_v(w_i) = \lambda_{v,i}w_i$ with $\lambda_{v,i} \neq 0$, for otherwise w_i commutes with the whole of $\mathfrak{h}(X)$, contradicts with the fact that $\mathfrak{h}(X)$ is a maximal abelian subalgebra of $\mathfrak{g}(X)$.

The Futaki invariant on a normal Fano orbifold is discussed in [DT]. Now note that the Futaki invariant \mathcal{F} vanishes on $[\mathfrak{g}(X), \mathfrak{g}(X)]$ as in the smooth case, we have

$$\mathcal{F}(w_i) = \lambda_{v,i}^{-1} \mathcal{F}([v, w_i]) = 0.$$

But a direct computation shows that up to a constant factor, $\mathcal{F}(v_i)$ is exactly $\int_P y^i dy$. The proposition follows from this fact. \square

4. EXISTENCE OF KÄHLER-RICCI SOLITONS

As in [WZ], we use the continuity method to consider a family of equations,

$$(4.1) \quad \det(g_{i\bar{j}}^0 + \varphi_{i\bar{j}}) = \det(g_{i\bar{j}}^0) e^{h - \theta_v - v(\varphi) - t\varphi}$$

with parameter $t \in [0, 1]$. Then ϕ satisfies the equation

$$(4.2) \quad \det(\phi_{i\bar{j}}) = e^{-c - w - \sum_i c_i \phi_i}$$

in \mathbb{R}^n , where

$$(4.3) \quad w = w_t := t\phi + (1 - t)\phi^0.$$

As in [WZ] and [TZ1], it suffices to obtain a uniform estimate for $\phi - \phi^0$ when $t \in [\varepsilon_0, 1]$.

The estimate is almost identical to that of [WZ], for readers' convenience, we include it briefly here.

Lemma 4.1. *Let $m_t := \inf_{x \in \mathbb{R}^n} w_t(x)$, then we have*

$$|m_t| \leq C$$

for some constant C independent of t .

Proof. The proof is the same to that of [WZ]. First, note that the image of the gradient map $D\phi$ is also the interior of the polytope P . By the equation (4.2) and the properties of Legendre transform, we have

$$(4.4) \quad \int_{\mathbb{R}^n} e^{-w} = \int_{\mathbb{R}^n} \det(\phi_{ij}) e^{c + \sum_i c_i \phi_i} dx = e^c \int_P e^{\sum_i c_i y^i} dy =: \beta.$$

Since $|Dw| \leq d_0 := \sup\{|y| \mid y \in P\}$, we have

$$\text{vol}(B_1(x^t)) e^{-m_t - d_0} \leq \beta,$$

thus $m_t \geq C$, for some constant C independent of t .

Next we derive the upper bound of m_t . Let $A_\lambda := \{x \in \mathbb{R}^n \mid w(x) \leq m_t + \lambda\}$. Then as in [WZ], we have $\text{vol}(A_1) \leq C e^{\frac{m_t}{2}}$. Then by convexity of w , we know that for any $\lambda > 1$ we have $\text{vol}(A_\lambda) \leq C \lambda^n e^{\frac{m_t}{2}}$, thus we can show that

$$\beta \leq C' e^{-\frac{m_t}{2}},$$

hence $m_t \leq C$. □

Lemma 4.2. *Let $x^t \in \mathbb{R}^n$ be the unique point such that $w_t(x^t) = m_t$, then we have*

$$|x^t| \leq C$$

for some constant C independent of t .

Proof. First note that $\text{vol}(A_1) \leq C$ by the proof of Lemma 4.1, and since $|Dw| \leq d_0$, there is a ball centered at x^t with fixed size contained in A_1 . If A_1 contains a point x with $|x - x^t|$ large, then by convexity of A_1 , the volume of A_1 will also be large. So we can choose a $R > 0$ independent of t such that $A_1 \subset B_R(x^t)$.

Also by convexity of w , we have

$$|Dw| > \frac{1}{R} \quad \text{in } \mathbb{R}^n \setminus B_R(x^t).$$

Hence for any $\varepsilon > 0$ small, we can find a sufficiently large R_ε (independent of t) such that

$$\int_{\mathbb{R}^n \setminus B_{R_\varepsilon}(x^t)} e^{-w} dx \leq \varepsilon.$$

On the other hand, for any $\varepsilon > 0$ small, we can find a large constant $C > 0$ such that if $|x^t| > C$, we have

$$\xi \cdot D\phi^0 > \frac{a_0}{2} \quad \text{in } B_{R_\varepsilon}(x^t),$$

where $\xi = x^t/|x^t|$, and $a_0 := \inf\{|y| \mid y \in \partial P\}$. Hence for ε sufficiently small, one has

$$\int_{\mathbb{R}^n} \xi \cdot D\phi^0 e^{-w} dx > 0.$$

However, by (3.3) and (4.2), we have

$$\begin{aligned} 0 &= \int_P y^i \exp(\sum_l c_l y^l) dy \\ &= \int_{\mathbb{R}^n} \phi_i \exp(\sum_l c_l \phi_l) \det D^2 \phi dx \\ &= e^c \int_{\mathbb{R}^n} \phi_i e^{-w} dx \\ &= -\frac{1-t}{t} e^c \int_{\mathbb{R}^n} \phi_i^0 e^{-w} dx. \end{aligned}$$

Thus

$$\int_{\mathbb{R}^n} \xi \cdot D\phi^0 e^{-w} dx = 0,$$

which is a contradiction. □

Proposition 4.3. *Let $\varphi = \varphi_t$, where $t \in [\varepsilon_0, 1]$, be a solution of (4.2), then*

$$\sup_X \varphi \leq C$$

for some constant C independent of t .

Proof. By Lemma 4.1 and Lemma 4.2, we know that $|w(0)| \leq C$, so $|\phi_0| \leq C$ for $t \in [\varepsilon_0, 1]$. From Lemma 2.2, we have a function v , whose graph is the asymptotical cone of the graph of ϕ^0 . Since $D\phi^0(\mathbb{R}^n) = D\phi(\mathbb{R}^n)$, we have

$$\phi(x) - \phi(0) \leq v(x) - v(0).$$

So we have

$$\varphi = \phi - \phi^0 \leq v - \phi^0 + \phi(0) - v(0).$$

Again by Lemma 2.2, we have $\sup_X \varphi = \sup_{\mathbb{R}^n} \varphi \leq C$. □

Next we need a Harnack type theorem to control the infimum of φ . Here we use an idea of Donaldson [Do], to prove it via the ordinary Sobolev imbedding theorem on P .

Proposition 4.4. *Let φ be as in Proposition 4.3, then we have*

$$\inf_X \varphi \geq -C$$

for some constant C independent of t .

Proof. Let the Legendre transform of ϕ be u . By definition, we have

$$u(y) = \sup_{\tilde{x} \in \mathbb{R}^n} (\tilde{x} \cdot y - \phi(\tilde{x})).$$

Then one can check easily that

$$\sup_{\mathbb{R}^n} (\phi^0 - \phi) = \sup_P (u - u^0).$$

Actually, suppose for $y \in P^o$, $x \in \mathbb{R}^n$ is the unique point such that $u(y) = x \cdot y - \phi(x)$, then we have

$$\begin{aligned} u(y) - u^0(y) &= x \cdot y - \phi(x) - \sup_{\tilde{x} \in \mathbb{R}^n} (\tilde{x} \cdot y - \phi^0(\tilde{x})) \\ &\leq x \cdot y - \phi(x) - x \cdot y + \phi^0(x) = \phi^0(x) - \phi(x) \\ &\leq \sup_{\mathbb{R}^n} (\phi^0 - \phi). \end{aligned}$$

Thus we get $\sup_P (u - u^0) \leq \sup_{\mathbb{R}^n} (\phi^0 - \phi)$, and the same argument implies that $\sup_{\mathbb{R}^n} (\phi^0 - \phi) \leq \sup_P (u - u^0)$.

Now it suffices to bound u on P .

The idea is to bound $\|Du\|_{L^p(P)}$ for $p > n$, then by the Sobolev embedding theorem on P , we get the estimate of $\text{osc}_P u$.

Note that

$$\int_P |Du|^p dy = \int_{\mathbb{R}^n} |x|^p \det(\phi_{ij}) dx \leq C \int_{\mathbb{R}^n} |x|^p e^{-w} dx.$$

Take R as in the proof of Lemma 4.2, then out of $B_R(x^t)$, we have

$$w(x) \geq m_t + 1 + \frac{1}{R}|x - x^t|,$$

thus

$$w(x) \geq \epsilon|x - x^t| - C \quad \text{in } \mathbb{R}^n$$

for some constants ϵ and C independent of t . Now it is obvious that we have

$$\|Du\|_{L^p(P)} \leq C.$$

Now by (7.45) of [GT], we have

$$\|u - u_P\|_{W^{1,p}} \leq C,$$

where $u_P := \frac{1}{\text{vol}(P)} \int_P u dy$ is the average of u over P . Then since the boundary of P is Lipschitz, we have the Sobolev imbedding

$$\sup_P |u - u_P| \leq C,$$

and hence

$$\text{osc}_P u \leq 2C.$$

So the proposition is true. \square

Proposition 4.3 and 4.4 complete the proof of Theorem 1.4.

Example 4.5. Let $Q \subset N_{\mathbb{R}} \cong \mathbb{R}^2$ be a lattice polytope, whose vertices are $(1, 0)$, $(0, 1)$ and $(-2, -1)$. Then the corresponding toric variety X_Q is a Fano orbifold with one singular point which is an ordinary double point. Actually, X_Q coincides with “A-1” in Nakagawa’s table on page 240 of [Na]. One can check easily that X_Q is a global quotient of \mathbb{CP}^2 . Note that the Fubini-Study metric descends to X_Q , but it is singular along a divisor. Actually, since the barycenter of the dual polytope P is not the origin, the Futaki invariant of X_Q is not zero, so X_Q does not admit a Kähler-Einstein metric. However, by Theorem 1.4, X_Q admits a Kähler-Ricci soliton metric.

Now we give an example of toric Fano variety with an invariant Kähler-Einstein metric, whose anticanonical divisor is not Cartier.

Example 4.6. Let $Q \subset N_{\mathbb{R}} \cong \mathbb{R}^2$ be a lattice polytope, whose vertices are $(-2, -1)$, $(-2, 1)$, $(2, -1)$ and $(2, 1)$. Then X_Q is a toric Fano orbifold. $-K_{X_Q}$ is not Cartier but $-2K_{X_Q}$ is. One can also check that X_Q is a global quotient of the surface “B-2” in Nakagawa’s table. The dual polytope of Q is

$$P = \{y \in M_{\mathbb{R}} \mid l_i(y) \geq 0, \ i = 1, 2, 3, 4\},$$

where $l_1(y) = -2y^1 - y^2 + 1$, $l_2(y) = -2y^1 + y^2 + 1$, $l_3(y) = 2y^1 - y^2 + 1$ and $l_4(y) = 2y^1 + y^2 + 1$. Obviously, the barycenter of P is the origin. By Theorem 1.4, X_Q admits a Kähler-Einstein metric.

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